16.7 Surface Integrals

Consider a Surface

S (smooth) : $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$(u,v) \in D.$

and a function $f(x,y,z)$ whose domain includes S.

We use the notation

$$\iint_S f(x,y,z)\, dS$$

for the surface integral of f over the surface S.

It can be shown (Power Point slides) that

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA}$$

Oriented Surfaces

Follow discussion on power-point slides.

- Mobius Strip: nonorientable surface.
- Orientable surface (two sided)
- At each point two unit vectors (orientable surface)

  $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$.

  It depends on how we perform the product

  $$\vec{n} = \vec{r}_u \times \vec{r}_v \quad \text{or} \quad -\vec{n} = \vec{r}_v \times \vec{r}_u .$$

- If $\vec{n}$ varies continuously over S, then S is called Oriented surface.
- Two possible orientation for an orientable surface

Show positive and negative orientation for sphere.

$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$$

$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi .$$

Then,

$$\vec{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{\|\vec{r}_\phi \times \vec{r}_\theta\|} = \frac{1}{a}\vec{r}(\theta,\phi) \quad \text{positive}$$

$$= \frac{1}{a}\langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle$$

$$= \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle.$$

<u>Convention</u>: Positive orientation for <u>closed Surfaces</u> is when normal vector points outward from Volume E enclosed by Surface S.

For example, at $(0,0,a)$
$\phi = 0$
$\Rightarrow \cos\phi = 1$
$\sin\phi = 0$
$\Rightarrow \hat{n} = \langle 0,0,1\rangle.$

<u>SURFACE INTEGRALS OF VECTOR FIELDS.</u>

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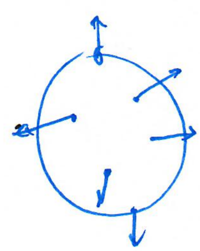
$$r_\phi \times r_\theta = \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi\rangle$$

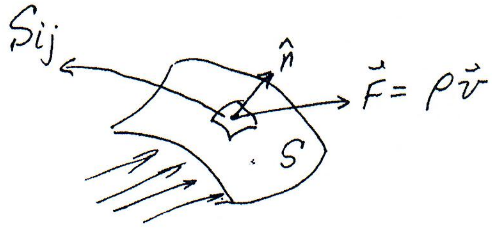
$$|r_\phi \times r_\theta| = a^2\sin\phi$$

$$\Rightarrow \hat{n} = \frac{r_\phi \times r_\theta}{|r_\phi \times r_\theta|} = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle$$



points outward.
positive orientation.
for closed surface.

Flux. SURFACE INTEGRAL OF VECTOR FIELDS.



Fluid with density $\rho(x,y,z)$ and velocity $\vec{v}(x,y,z)$ flowing through S

$$[\rho] = \frac{M}{Vol} = \frac{M}{L^3}, \qquad [\vec{v}] = \frac{L}{t}$$

$$\Rightarrow \quad [\rho\vec{v}] = \frac{M}{L^3}\cdot\frac{L}{t} = \frac{M}{t}\cdot\frac{1}{L^2}$$

So $\rho\vec{v}$ is the rate of flow or mass per unit of time, per unit of area

Therefore, the amount of mass of fluid crossing S_{ij} (Small patch of S) per unit of time can be approximated by

$$(\rho\vec{v}\cdot\hat{n})\,\Delta S_{ij}$$

evaluated at a point in S_{ij}

Remark: It is the normal component of the velocity, the one responsable to carry fluid across S_{ij}.

By adding the fluid mass across every S_{ij} we get

$$\begin{array}{c}\text{Total fluid mass} \\ \text{across } S \text{ per unit of} \\ \text{time}\end{array} = \begin{array}{c}\text{Flux of} \\ \text{mass} \\ \text{across} \\ S\end{array} = \iint_S (\rho\vec{v}).\underbrace{\hat{n}}_{\text{Unit normal}}\, ds$$

In General, we can define a flux integral for Vector fields across Surfaces S satisfying the properties specified below:

<u>Definition</u>:

If $\vec{F}$ is a continuous Vector field defined on an oriented Surface S, with unit normal vector $\hat{n}$, then

$$\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}\, ds$$

is called the surface integral of $\vec{F}$ over S, or the flux of $\vec{F}$ across S.

<u>Rmk</u>: There are two possibilities for $\iint_S \vec{F}.\hat{n}ds$ depending on direction of $\hat{n}$. These are

$$\pm\iint_S \vec{F}.\hat{n}\, ds.$$

For S given in parametric vector form

$$S: \vec{r}(u,v), \qquad (u,v) \in D.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} dS$$

$$= \iint_S \vec{F}(r(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, dA$$

$$= \iint_S \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA$$

Exercise (Not in book)

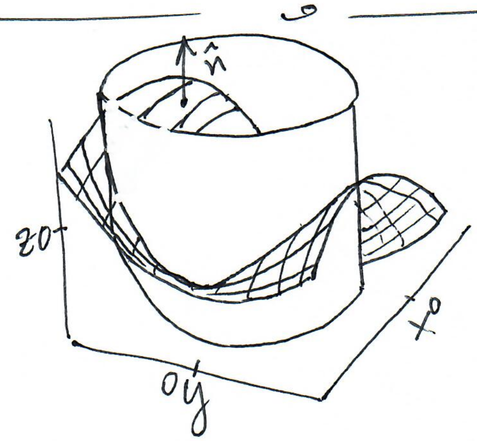
Evaluate the Surface integral

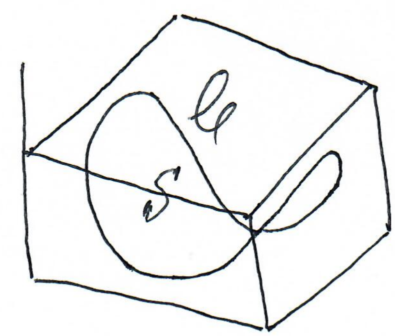
$$\iint_S \vec{G}\cdot d\vec{S}$$

where $\vec{G}(x,y,z) = \langle x, -y, 0\rangle = x\hat{\imath} - y\hat{\jmath} + 0\hat{k}$

and S is the upward oriented surface part of the hyperbolic paraboloid: $z = y^2 - x^2$ bounded by the closed curve $\mathcal{C}$ obtained from the intersection of $\begin{cases} S_1: z = y^2 - x^2 \\ S_2: x^2 + y^2 = 1. \end{cases}$

Answ:





Parametric representation of :

$\vec{r}(u,v)$: $x(u,v) = u$, $y(u,v) = v$, $z(u,v) = v^2 - u^2$

$(u,v) \in D$

$D: \{(u,v) : u^2 + v^2 \leq 1\}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = \langle 2u, -2v, 1 \rangle$$

upward normal

Then,

$$\iint_S \vec{G}(s)\,d\vec{s} = \iint_D \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\,dA$$

$$= \iint_{D} \langle u, -v, 0 \rangle \cdot \langle 2u, -2v, 1 \rangle \, dA$$

D Disk of radius $=1$

$$= \iint_D (2u^2 + 2v^2)\,dA = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (2u^2 + 2v^2)\,dv\,du$$

Cylind. coords $\longrightarrow$ $u = r\cos\theta$, $v = r\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$= \int_0^{2\pi} \int_0^1 (2r^2\cos^2\theta + 2r^2\sin^2\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r^2(\underbrace{\cos^2 + \sin^2}_{=1})\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \frac{1}{2}2\pi = \underline{\underline{\pi}}$$